# 第七章
# 傅立葉級數與變換

## 習題 7-1

求下列各函數的傅立葉級數.

**2.** $f(x)=\begin{cases} 0\ , & -\pi<x<0 \\ x^2, & 0<x<\pi \end{cases}$

解：$a_0=\dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx=\dfrac{1}{\pi}\int_0^{\pi} x^2\,dx=\dfrac{\pi^2}{3}$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx=\frac{2\cos n\pi}{n^2}=\frac{2(-1)^n}{n^2}$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx$$

$$=-\frac{\pi\cos n\pi}{n}+\frac{2}{\pi n^3}(\cos n\pi-1)=\frac{2[(-1)^n-1]}{\pi n^3}-\frac{\pi(-1)^n}{n}$$

故所求的傅立葉級數爲

$$f(x)=\frac{\pi^2}{6}+\sum_{n=1}^{\infty}\left\{\frac{2(-1)^n}{n^2}\cos nx+\frac{2[(-1)^n-1]-\pi^2 n^2(-1)^n}{\pi n^3}\sin nx\right\}$$

**6.** $f(x)=\begin{cases} 1, & 0<x<\dfrac{\pi}{2} \\ 0, & \dfrac{\pi}{2}<x<2\pi \end{cases}$

解：$a_0=\dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\,dx=\dfrac{1}{2}$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi/2}\cos nx\,dx=\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi/2}\sin nx\,dx=\frac{1}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

故所求的傅立葉級數爲

$$f(x)=\frac{1}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\sin\frac{n\pi}{2}\cos nx+\left(1-\cos\frac{n\pi}{2}\sin nx\right)\right]$$

## 習題 7-2

求 2～3 題中各函數的半幅正弦級數.

**2.** $f(x)=e^x$，$0<x<1$

解：$$b_n=\frac{2}{1}\int_0^1 e^x\sin n\pi x\,dx=\frac{2n\pi(1-e\cos n\pi)}{1+n^2\pi^2}$$

故 $$f(x)=2\pi\sum_{n=1}^{\infty}\frac{n(1-e\cos n\pi)}{n^2\pi^2+1}\sin n\pi x$$

求 4～6 題中各函數的半幅餘弦級數.

**4.** $f(x)=x^2$，$0<x<2$

解：$$a_0=\frac{2}{2}\int_0^2 f(x)\,dx=\int_0^2 x^2\,dx=\frac{8}{3}$$

$$a_n=\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx=\int_0^2 x^2\cos\frac{n\pi x}{2}\,dx$$

$$=\frac{2}{n\pi}\left(x^2\sin\frac{n\pi x}{2}\Big|_0^2-2\int_0^2 x\sin\frac{n\pi x}{2}\,dx\right)$$

$$=\frac{16}{n^2\pi^2}\cos n\pi$$

$$=(-1)^n \frac{16}{n^2\pi^2}$$

故 $$f(x)=\frac{4}{3}+\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}$$

**6.** $f(x)=\cos x$，$0<x<2$

**解**：$$a_0=\frac{2}{2}\int_0^2 f(x)\,dx=\int_0^2\cos x\,dx=\sin 2$$

$$a_n=\frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx=\int_0^2\cos x\cos\frac{n\pi x}{2}\,dx$$

$$=\frac{1}{2}\int_0^2\left[\cos\left(1+\frac{n\pi}{2}\right)x+\cos\left(1-\frac{n\pi}{2}\right)x\right]dx$$

$$=\frac{1}{2}\left[\left.\frac{\sin\left(1+\frac{n\pi}{2}\right)x}{1+\frac{n\pi}{2}}\right|_0^2+\left.\frac{\sin\left(1-\frac{n\pi}{2}\right)x}{1-\frac{n\pi}{2}}\right|_0^2\right]$$

$$=\frac{1}{2}\left[\frac{\sin(2+n\pi)}{1+\frac{n\pi}{2}}+\frac{\sin(2-n\pi)}{1-\frac{n\pi}{2}}\right]$$

$$=\sin 2\cos n\pi\left(\frac{1}{2+n\pi}+\frac{1}{2-n\pi}\right)$$

$$=\frac{4(-1)^n\sin 2}{4-n^2\pi^2}$$

故 $$f(x)=\frac{\sin 2}{2}+4\sin 2\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{4-n^2\pi^2}\cos\frac{n\pi x}{2}\right]$$

## 習題 7-3

討論 1～2 題中各系統的穩態運動 (見圖 7-3-1).

**1.** $F(t)=t$，$-\frac{1}{2}<t<\frac{1}{2}$，$F(t+1)=F(t)$

$k=40$ 克/秒$^2$，$m=100$ 克，$c=0.1$.

**解**：$b_n=\frac{4}{1}\int_0^{1/2} t\sin 2n\pi t\,dt=-\frac{\cos n\pi}{n\pi}=\frac{(-1)^{n+1}}{n\pi}$

$$F(t)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin 2n\pi t$$

$$=\frac{1}{\pi}\left(\sin 2\pi t-\frac{1}{2}\sin 4\pi t+\frac{1}{3}\sin 6\pi t-\cdots\right)$$

考慮微分方程式

$$100\frac{d^2y}{dt^2}+0.1\frac{dy}{dt}+40y=F(t)=\frac{(-1)^{n+1}}{n\pi}\sin 2n\pi t,\ n=1,2,3,\cdots$$

上式於穩態狀況下的解為

$$y_n(t)=A_n\cos 2n\pi t+B_n\sin 2n\pi t,\ n=1,2,3,\cdots$$

將此解代入該微分方程式中，可得

$$A_n=\frac{0.2(-1)^n}{(400n^2\pi^2-40)^2+(0.2n\pi)^2}=\frac{0.2(-1)^n}{D_n}$$

$$B_n=\frac{(400n^2\pi^2-40)(-1)^n}{n\pi[(400n^2\pi^2-40)^2+(0.2n\pi)^2]}=\frac{(-1)^n}{n\pi D_n}(400n^2\pi^2-40)$$

其中 $D_n=(400n^2\pi^2-40)^2+(0.2n\pi)^2$

故

$$y_n(t)=\frac{0.2(-1)^n}{D_n}\cos 2n\pi t+\frac{40(-1)^n(10n^2\pi^2-1)}{n\pi D_n}\sin 2n\pi t,\ n=1,2,3,\cdots$$

討論 3～4 題中各電路的穩態電流.

**3.** $E(t)=100\sin 50\pi t$，$0\leqslant t\leqslant 0.02$，$E(t+0.02)=E(t)$

解：$a_0=\dfrac{1}{0.01}\displaystyle\int_0^{0.02} 100\sin 50\pi t\,dt=\dfrac{400}{\pi}$

$$a_n=\frac{1}{0.01}\int_0^{0.02} 100\sin 50\pi t\cos 100n\pi t\,dt$$

$$=5000\int_0^{0.02}[\sin 50\pi(1+2n)t+\sin 50\pi(1-2n)t]\,dt$$

$$=5000\left[-\frac{\cos 50\pi(1+2n)t}{50\pi(1+2n)}\Bigg|_0^{0.02}-\frac{\cos 50\pi(1-2n)t}{50\pi(1-2n)}\Bigg|_0^{0.02}\right]$$

$$=\frac{400}{\pi}\left(\frac{1}{1-4n^2}\right)$$

$$b_n=\frac{1}{0.01}\int_0^{0.02} 100\sin 50\pi t\sin 100n\pi t\,dt$$

$$=5000\left[\frac{\sin 50\pi(1-2n)t}{50\pi(1-2n)}\Bigg|_0^{0.02}-\frac{\sin 50\pi(1+2n)t}{50\pi(1+2n)}\Bigg|_0^{0.02}\right]=0$$

$$E(t)=\frac{200}{\pi}+\sum_{n=1}^{\infty}\frac{400}{\pi(1-4n^2)}\cos 100\,n\pi t$$

考慮微分方程式

$$0.4\frac{d^2I}{dt^2}+100\frac{dI}{dt}+10^5I=\frac{40000n}{4n^2-1}\sin 100n\pi t，n=1，2，3，\cdots$$

上式於穩態狀況下的解爲

$$I_n(t)=A_n\cos 100n\pi t+B_n\sin 100n\pi t，n=1，2，3，\cdots$$

代入微分方程式可得

$$\begin{cases}(50-2n^2\pi^2)A_n+5n\pi\,B_n=0\\(50-2n^2\pi^2)B_n-5n\pi\,A_n=\dfrac{20n}{4n^2-1}\end{cases}$$

$$故\begin{cases} A_n=\dfrac{100n^2\pi}{(1-4n^2)(D_n^2+25n^2\pi^2)} \\ \\ B_n=\dfrac{20nD_n}{(4n^2-1)(D_n^2+25n^2\pi^2)} \end{cases}，其中\ D_n=2(25-n^2\pi^2)$$

## 習題 7-4

**1.** 試以傅立葉積分將下列各函數展開.

(1) $f(x)=\begin{cases} 0, & |t|>\pi \\ t, & -\pi\leqslant t\leqslant\pi \end{cases}$

**解**：(1) 由於 $\displaystyle\int_{-\infty}^{\infty}|f(t)|\,dt=\int_{-\infty}^{\infty}|t|\,dt=2\int_0^{\pi}t\,dt=\pi^2$

故傅立葉積分式可代表 $f(t)$.

$$A(\lambda)=\int_{-\infty}^{\infty}t\cos(\lambda t)\,dt=0\quad(\because t\cos(\lambda t)\ 是奇函數)$$

且 $\displaystyle B(\lambda)=\int_{-\infty}^{\infty}t\sin(\lambda t)\,dt=\int_{-\pi}^{\pi}t\sin(\lambda t)\,dt=2\int_0^{\pi}t\sin(\lambda t)\,dt$

$$=2\left[-\frac{t}{\lambda}\cos(\lambda t)+\frac{\sin(\lambda t)}{\lambda^2}\bigg|_0^{\pi}\right]$$

$$=2\left[\frac{\sin(\lambda\pi)}{\lambda^2}-\frac{\pi}{\lambda}\cos(\lambda\pi)\right]$$

$\therefore f$ 之傅立葉積分表示式為

$$f(t)=\frac{1}{\pi}\int_0^{\infty}B(\lambda)\sin(\lambda t)\,d\lambda$$

$$=\frac{1}{\pi}\int_0^{\infty}\left[\frac{2\sin(\lambda\pi)}{\lambda^2}-\frac{2\pi}{\lambda}\cos(\lambda\pi)\right]\sin(\lambda t)\,d\lambda$$

$$=\int_0^{\infty}\left[\frac{2\sin(\lambda\pi)}{\pi\lambda^2}-\frac{2}{\lambda}\cos(\lambda\pi)\right]\sin(\lambda t)\,d\lambda$$

$$=\begin{cases}-\dfrac{\pi}{2} & ，若\ t=-\pi\\ t & ，若\ -\pi<t<\pi\\ \dfrac{\pi}{2} & ，若\ t=\pi\\ 0 & ，若\ |t|>\pi\end{cases}$$

**2.** 設函數 $f(t)=\begin{cases}1，0<t<1\\0，t>1\end{cases}$，試求 (1) 傅立葉正弦積分，(2) 傅立葉餘弦積分.

**解**：(1) 傅立葉正弦積分

$$f(t)=\frac{1}{\pi}\int_0^{\infty}B(\lambda)\sin(\lambda t)\,d\lambda$$

其中 $B(\lambda)=2\int_0^{\infty}f(u)\sin(\lambda u)\,du$

$$B(\lambda)=2\int_0^{1}1\sin(\lambda u)\,du=2\left(-\frac{1}{\lambda}\cos\lambda u\,\Big|_0^1\right)$$

$$=-\frac{2}{\lambda}(\cos\lambda-1)=\frac{2(1-\cos\lambda)}{\lambda}$$

$$\therefore f(t)=\frac{1}{\pi}\int_0^{\infty}\frac{2(1-\cos\lambda)}{\lambda}\sin(\lambda t)\,d\lambda=\frac{2}{\pi}\int_0^{\infty}\frac{1-\cos\lambda}{\lambda}\sin(\lambda t)\,d\lambda$$

(2) 傅立葉餘弦積分

$$f(t)=\frac{1}{\pi}\int_0^{\infty}A(\lambda)\cos(\lambda t)\,d\lambda$$

其中 $A(\lambda)=2\int_0^{\infty}f(u)\cos(\lambda u)\,du$

$$A(\lambda)=2\int_0^1 1\cos(\lambda u)\,du=2\left[\frac{1}{\lambda}\sin(\lambda u)\right]_0^1=\frac{2\sin\lambda}{\lambda}$$

$$\therefore f(t)=\frac{2}{\pi}\int_0^\infty \frac{\sin\lambda}{\lambda}\cos(\lambda t)\,d\lambda$$

**4.** 試求函數 $f(t)$ 之傅立葉餘弦積分

$$f(t)=\begin{cases}2t, & 0<t<1\\ 0, & t>1\end{cases}$$

**解**：因 $f(t)=\dfrac{1}{\pi}\displaystyle\int_0^\infty A(\lambda)\cos(\lambda t)\,d\lambda$

其中 $A(\lambda)=2\displaystyle\int_0^\infty f(u)\cos(\lambda u)\,du$

$$A(\lambda)=2\int_0^1 2u\cos(\lambda u)\,du=4\int_0^1 u\cos(\lambda u)\,du$$

令 $u'=u$，$dv'=\cos(\lambda u)\,du$，則 $du'=du$，$v'=\dfrac{\sin(\lambda u)}{\lambda}$

故 $\displaystyle\int_0^1 u\cos(\lambda u)\,du=\frac{u}{\lambda}\sin(\lambda u)\Big|_0^1-\int_0^1\frac{1}{\lambda}\sin(\lambda u)\,du$

$$=\frac{\sin\lambda}{\lambda}-\left[-\frac{1}{\lambda^2}\cos(\lambda u)\Big|_0^1\right]$$

$$=\frac{\sin\lambda}{\lambda}+\left(\frac{\cos\lambda}{\lambda^2}-\frac{1}{\lambda^2}\right)$$

$$=\frac{\sin\lambda}{\lambda}+\frac{\cos\lambda-1}{\lambda^2}$$

$$\therefore A(\lambda)=4\left[\frac{\sin\lambda}{\lambda}+\frac{\cos\lambda-1}{\lambda^2}\right]$$

故 $$f(t)=\frac{4}{\pi}\int_0^\infty\left[\frac{\sin\lambda}{\lambda}+\frac{\cos\lambda-1}{\lambda^2}\right]\cos(\lambda t)\,d\lambda$$

**5.** 試利用傅立葉積分式證明

$$\int_0^\infty \frac{\cos \lambda t}{9+\lambda^2}\, d\lambda = \frac{\pi}{6} e^{-3t} \quad (t>0)$$

**解**：令 $f(t)=e^{-3t}$ $(t>0)$

如果 $f(t)$ 爲偶函數，則 $B(\lambda)=0$，而

$$A(\lambda)=\frac{2}{\pi}\int_0^\infty f(u)\cos(\lambda u)\, du=\frac{2}{\pi}\int_0^\infty e^{-3u}\cos(\lambda u)\, du$$

$$=\frac{2}{\pi}\cdot\frac{1}{9+\lambda^2}\left[e^{-3u}(\lambda \sin \lambda u-3\cos \lambda u)\right]_0^\infty$$

$$=\frac{2}{\pi}\,\frac{3}{9+\lambda^2}$$

$$\therefore f(t)=e^{-3t}=\frac{2}{\pi}\int_0^\infty \frac{3}{9+\lambda^2}\cos \lambda t\, d\lambda$$

於是
$$\int_0^\infty \frac{\cos \lambda t}{9+\lambda^2}\, d\lambda=\frac{\pi}{6} e^{-3t} \quad (t>0)$$

## 習題 7-5

**1.** 已知 $f(x)=\begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$

(1) 試求傅立葉積分式.　　(2) 利用 (1) 之結果求 $\displaystyle\int_0^\infty \frac{\sin \lambda}{\lambda}\, d\lambda$.

(3) 利用 (1) 之結果求 $\displaystyle\int_0^\infty \frac{\sin \lambda \cos \lambda}{\lambda}\, d\lambda$.

**解**：(1) 因 $f(x)$ 爲偶函數，利用傅立葉餘弦積分式，得

$$f(x)=\frac{1}{\pi}\int_0^\infty\left[2\int_0^\infty f(u)\cos(\lambda u)\,du\right]\cos(\lambda x)\,d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^\infty f(u)\cos(\lambda u)\,du\right]\cos(\lambda x)\,d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\left[\int_0^1 \cos(\lambda u)\,du\right]\cos(\lambda x)\,d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda}{\lambda}\cos(\lambda x)\,d\lambda$$

$$=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda\cos(\lambda x)}{\lambda}\,d\lambda$$

(2) 令 $x=0$ 代入上式，得

$$f(0)=1=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda}{\lambda}\,d\lambda$$

亦即 $$\int_0^\infty\frac{\sin\lambda}{\lambda}\,d\lambda=\frac{\pi}{2}$$

(3) 令 $x=1$ 代入傅立葉餘弦積分式，得

$$f(1)=\frac{1}{2}=\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda\cos\lambda}{\lambda}\,d\lambda$$

故 $$\int_0^\infty\frac{\sin\lambda\cos\lambda}{\lambda}\,d\lambda=\frac{\pi}{4}$$

**2.** 試求下列各函數之複數傅立葉積分.

(1) $f(x)=xe^{-|x|}$

**解**：(1) 首先$\displaystyle\int_{-\infty}^\infty |f(x)|\,dx=2\int_0^\infty xe^{-x}\,dx=2\lim_{t\to\infty}\int_0^t xe^{-x}\,dx$

$$=2\lim_{t\to\infty}\left(-xe^{-x}-e^{-x}\Big|_0^t\right)=2$$

故 $f(x)$ 爲絕對收斂.

$$C(\lambda)=\frac{1}{2}\int_{-\infty}^{\infty} ue^{-|u|}\,e^{-i\lambda u}\,du$$

$$=\frac{1}{2}\left[\int_{-\infty}^{0} ue^{u(1-i\lambda)}\,du+\int_{0}^{\infty} ue^{-u(1+i\lambda)}\,du\right]$$

$$=\frac{1}{2}\left[\frac{1}{(1+i\lambda)^2}-\frac{1}{(1-i\lambda)^2}\right]$$

$$=\frac{-2i\lambda}{(1+\lambda^2)^2}$$

則 $f(x)=xe^{-|x|}$ 之複數傅立葉積分爲

$$\frac{-2i}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{(1+\lambda^2)^2}\,e^{i\lambda x}\,d\lambda$$

由於 $f(x)$ 對所有 $x$ 皆連續，故得

$$f(x)=xe^{-|x|}=-\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{\lambda}{(1+\lambda^2)^2}\,e^{i\lambda x}\,d\lambda$$

**3.** 求下式 $f(t)$ 之傅立葉正弦積分式

$$f(t)=\begin{cases} t, & -\pi\leqslant t\leqslant\pi \\ 0, & t>\pi \end{cases}$$

**解**：由傅立葉正弦積分式

$$f(t)=\frac{2}{\pi}\int_{0}^{\infty}\left[\int_{0}^{\infty} f(u)\sin\lambda u\,du\right]\sin(\lambda t)\,d\lambda$$

可求得

$$\int_{0}^{\infty} f(u)\sin\lambda u\,du=\int_{0}^{\pi} f(u)\sin\lambda u\,du+\int_{\pi}^{\infty} f(u)\sin\lambda u\,du$$

$$=\int_0^{\pi} u \sin \lambda u \, du$$

$$=\left(\frac{\sin \lambda u}{\lambda^2}-\frac{u \cos \lambda u}{\lambda}\right)\Bigg|_0^{\pi}$$

$$=\frac{\sin \lambda\pi}{\lambda^2}-\frac{\pi \cos \lambda\pi}{\lambda}$$

因此，

$$f(t)=\frac{2}{\pi}\int_0^{\infty}\left(\frac{\sin \lambda\pi}{\lambda^2}-\frac{\pi \cos \lambda\pi}{\lambda}\right)\sin(\lambda t)\,d\lambda$$

**4.** 試求下列各函數之傅立葉正弦及傅立葉餘弦變換**.**

(1) $f(t)=e^{-t}$

**解**：(1) $F_s(\omega)=\displaystyle\int_0^{\infty} e^{-t}\sin(\omega t)\,dt=\lim_{b\to\infty}\int_0^{b} e^{-t}\sin(\omega t)\,dt=\frac{\omega}{1+\omega^2}$

$$F_c(\omega)=\int_0^{\infty} e^{-t}\cos(\omega t)\,dt=\lim_{b\to\infty}\int_0^{b} e^{-t}\cos(\omega t)\,dt=\frac{1}{1+\omega^2}$$

**5.** 試求下列各函數之逆傅立葉變換**.**

(1) $\dfrac{e^{(24-4\omega)i}}{4-(6-\omega)i}$

**解**：(1) $F(\omega)=\dfrac{e^{(24-4\omega)i}}{4-(6-\omega)i}=\dfrac{e^{-4(\omega-6)i}}{4+(\omega-6)i}$

故 $f(t)=\mathcal{F}^{-1}\{F(\omega)\}=e^{6it}\,\mathcal{F}^{-1}\left\{\dfrac{e^{-4i\omega}}{4+\omega i}\right\}=e^{6it}\,u(t-4)e^{-4(t-4)}$

**7.** 試求微分方程式 $y'-2y=u(t)\,e^{-2t}$ $(t\in\mathbb{R})$ 之特解**.**

**解**：令 $\mathcal{F}\{y(t)\}=Y(\omega)$，則得

$$\mathcal{F}\{y'-2y\}=\mathcal{F}\{u(t)\,e^{-2t}\}$$

$$\mathcal{F}\{y'\}-2\mathcal{F}(y)=\frac{1}{2+i\omega}$$

故
$$i\omega\ Y(\omega)-2Y(\omega)=\frac{1}{2+i\omega}$$

解得
$$Y(\omega)=\frac{1}{(2+i\omega)(i\omega-2)}=-\frac{1}{4+\omega^2}$$

故　$y(t)=\mathcal{F}^{-1}\left\{-\dfrac{1}{4+\omega^2}\right\}=-\dfrac{1}{4}\ \mathcal{F}^{-1}\left\{\dfrac{2\cdot 2}{2^2+\omega^2}\right\}=-\dfrac{1}{4}\ e^{-2|t|}$

或
$$y(t)=\begin{cases}-\dfrac{1}{4}\ e^{2t}\quad, & t<0\\[2mm] -\dfrac{1}{4}\ e^{-2t}, & t\geqslant 0\end{cases}.$$